Lecture 9: Exact Puppe sequence

Definition

A sequence of maps of sets with base points (i.e. in $\mathbf{Set}_*$)

$$(A, a_0) \xrightarrow{f} (B, b_0) \xrightarrow{g} (C, c_0)$$

is said to be exact at B if $\mathrm{im}(f) = \ker(g)$ where

$$\mathrm{im}(f) = f(A), \quad \ker(g) = g^{-1}(c_0).$$

A sequence

$$\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$$

is called an exact sequence if it is exact at every A_i.

Example

Let $H \lhd G$ be a normal subgroup. There is a short exact sequence

$$1 \to H \to G \to G/H \to 1$$

in **Group**. Here we view **Group** as a subcategory of $\mathbf{Set}_*$ where a group is based at its identity element.

Definition

A sequence of maps in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

is called exact if for any $Y \in \underline{\mathrm{h}\mathscr{T}_\star}$, the following sequence of pointed sets is exact

$$\cdots \to [Y, X_{n+1}]_0 \to [Y, X_n]_0 \to [Y, X_{n-1}]_0 \to \cdots$$

Definition

Let $f\colon (X, x_0) \to (Y, y_0)$ be a map in $\mathscr{T}_\star$. We define its homotopy fiber F_f in $\mathscr{T}_\star$ by the pull-back diagram

$$\begin{array}{ccc} F_f & \longrightarrow & P_{y_0}Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

$$F_f = \{(x, \gamma) \in X \times PY | \gamma(0) = y_0, \gamma(1) = f(x)\}$$

Recall that $p_1 : P_{y_0}Y \to Y$ is a fibration, thus

Lemma

$\pi : F_f \to X$ is a fibration.

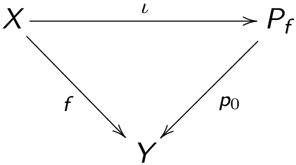
Note that F_f is precisely the fiber of $P_f \to Y$ over y_0:

$$\begin{array}{ccc} F_f & \longrightarrow & P_f \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ y_0 & \hookrightarrow & Y. \end{array}$$

So this is the same as our definition before. We will put extra emphasize on the role of based point in this section.

Lemma

If $f\colon X \to Y$ is a fibration, then $f^{-1}(y_0)$ is homotopy equivalent to its homotopy fiber F_f.



For arbitrary map $f: X \to Y$, we still have a canonical map

$$j: f^{-1}(y_0) \to F_f$$

which may not be a homotopy equivalence. The homotopy fiber can be viewed as a good replacement of fiber in homotopy category that behaves nicely for fibrations.

Lemma

The sequence

$$F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is exact at X in $\underline{\mathrm{h}\mathscr{T}_\star}$.

Proof: Let y_0 be the base point of Y. We first observe that $f \circ \pi$ factors through $P_{y_0} Y$ which is contractible. Therefore $f \circ \pi$ is null homotopy. Let $Z \in \underline{\mathrm{h}\mathscr{T}_\star}$. Consider

$$[Z, F_f]_0 \xrightarrow{\pi_*} [Z, X]_0 \xrightarrow{f_*} [Z, Y]_0.$$

Since $f \circ \pi$ is null homotopic, we have $\mathrm{im}\, \pi_* \subset \ker f_*$.

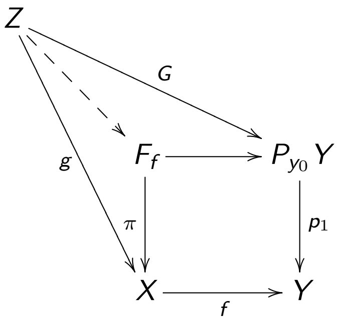
Let $g : Z \to X$, $[g]_0 \in \ker f_*$. Assume $f \circ g$ is based homotopic to the trivial map by

$$G\colon Z \times I \to Y.$$

Since $G|_{Z\times\{0\}} = y_0$, it can be regarded as a map

$$G\colon Z \to P_{y_0}Y$$

that fits into the following diagram



Therefore the pair (G, g) factors through F_f. So $[g]_0 \in \operatorname{im} \pi_*$.

Notice that the fiber of F_f over x_0 is precisely ΩY

$$\begin{array}{ccc} \Omega Y & \longrightarrow & F_f \\ \downarrow & & \downarrow \pi \\ x_0 & \hookrightarrow & X. \end{array}$$

We find the following sequence of pointed maps

$$\Omega X \xrightarrow{\Omega f} \Omega Y \to F_f \xrightarrow{\pi} X \xrightarrow{f} Y.$$

Lemma

The sequence $\Omega X \xrightarrow{\Omega f} \Omega Y \to F_f \xrightarrow{\pi} X \xrightarrow{f} Y$ is exact in $\underline{\mathrm{h}\mathscr{T}_\star}$.

Proof: We will construct the following commutative diagram in $\underline{\mathrm{h}\mathscr{T}_\star}$ with all vertical arrows homotopy equivalences

$$\begin{array}{ccccccccc}
\Omega X & \xrightarrow{\Omega f} & \Omega Y & \longrightarrow & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \tilde{j}'} & & \downarrow{\scriptstyle j} & & \downarrow & & \downarrow & & \downarrow \\
F_{\pi'} & \xrightarrow{\pi''} & F_\pi & \xrightarrow{\pi'} & F_f & \xrightarrow{\pi} & X & \xrightarrow{f} & Y
\end{array}$$

F_π is the homotopy fiber of $\pi : F_f \to X$, given by the pull-back

$$\begin{array}{ccc} F_\pi & \longrightarrow & P_{x_0}X \\ \downarrow & & \downarrow p_1 \\ F_f & \xrightarrow{\pi} & X \end{array}$$

or explicitly

$$F_\pi = \{([\gamma], [\beta]) \in P_{y_0}Y \times P_{x_0}X | f(\beta(1)) = \gamma(1)\}.$$

Since $F_f \xrightarrow{\pi} X$ is a fibration with fiber ΩY, the map

$$j : \Omega Y \to F_\pi, \quad j([\beta]) = ([1_{x_0}], [\beta])$$

is the natural map of fiber into homotopy fiber which is a homotopy equivalence. This gives the second square.

Similarly, the fiber of the fibration $F_\pi \to F_f$ is ΩX. We find

$$j' : \Omega X \to F_{\pi'}$$

from fiber to homotopy fiber, which is a homotopy equivalence. Let

$$(-)^{-1} : \Omega X \to \Omega X, \quad \gamma \to \gamma^{-1}$$

be the inverse of the loop. We define

$$\tilde{j}' = j' \circ (-)^{-1} : \Omega X \to F_{\pi'}.$$

We define the map $k : \Omega X \to F_\pi$ by the commutative diagram

$$\begin{array}{ccc} \Omega X & & \\ {\scriptstyle \tilde{j}'} \downarrow & \overset{k}{\searrow} & \\ F_{\pi'} & \xrightarrow[\pi'']{} & F_\pi \end{array}$$

Consider the diagram

$$\begin{array}{ccc} \Omega X & \xrightarrow{\Omega f} & \Omega Y \\ & \searrow^{k} & \downarrow j \\ & & F_\pi . \end{array}$$

This diagram is NOT commutative in $\mathscr{T}_\star$.
However, $j \circ \Omega f$ is homotopic to k, so this diagram commutes in $\mathrm{h}\mathscr{T}_\star$. To see this, let us explicitly write

$$k([\gamma]) = ([\gamma^{-1}], [1_{y_0}]), \quad \big(j \circ \Omega f\big)(\gamma) = ([1_{x_0}], [f(\gamma)]).$$

They are homotopic via

$$F([\gamma], t) = \Big([(\gamma\,|_{[t,1]})^{-1}], f\big[\gamma\,|_{[0,t]}\big] \Big).$$

So the first square commutes in $\mathrm{h}\mathscr{T}_\star$. The lemma follows. □

Lemma

Let $X_1 \to X_2 \to X_3$ be exact in $\underline{\mathrm{h}\mathscr{T}_\star}$, then so is

$$\Omega X_1 \to \Omega X_2 \to \Omega X_3.$$

Proof.

For any Y, apply $[Y, -]_0$ to the exact sequence $X_1 \to X_2 \to X_3$ and use the fact that Ω is right adjoint to the suspension Σ, i.e.

$$[\Sigma Y, X_i]_0 = [Y, \Omega X_i]_0,$$

we obtain an exact sequence. This implies the lemma. □

Theorem (Exact Puppe Sequence)

Let $f\colon X \to Y$ in $\underline{\mathscr{T}_\star}$. Then the following sequence in exact in $\underline{\mathrm{h}\mathscr{T}_\star}$

$$\cdots \to \Omega^2 Y \to \Omega F_f \to \Omega X \to \Omega Y \to F_f \to X \to Y.$$

Theorem

Let $p : E \to B$ be a map in $\underline{\mathscr{T}_\star}$. Assume p is a fibration whose fiber over the base point is F. Then we have an exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to \pi_0(B).$$

Proof: Since p is a fibration, F is homotopy equivalent to F_p. Observe that

$$[S^0, \Omega^n X]_0 = [\Sigma^n S_0, X]_0 = [S^n, X] = \pi_n(X).$$

The theorem follows by applying $[S^0, -]_0$ to the Puppe Sequence associated to $p : E \to B$. $\square$

Example

Consider the universal cover $\exp : \mathbb{R}^1 \to S^1$. The associated long exact sequence implies

$$\pi_n(S^1) = 0, \quad \forall n > 1.$$

Proposition

If $i < n$, then $\pi_i(S^n) = 0$.

Proof.

Let $f\colon S^i \to S^n$. We need the following fact: any continuous map from a compact smooth manifold X to S^n can be uniformly approximated by a smooth map. Furthermore, two smooth maps are continuously homotopic, then they are smoothly homotopic.

Thus, we can assume that f is homotopic to a smooth map f'. Then f' is not surjective (for dimension reason). Thus, $f' : S^i \to (S^n - \{\mathrm{pt}\}) \simeq \mathbb{R}^n \simeq \{\mathrm{pt}\}$ is null homotopic. □

Example

Consider the Hopf fibration $S^3 \to S^2$ with fiber S^1. The associated long exact sequence of homotopy groups implies

$$\pi_2(S^2) \simeq \mathbb{Z}, \quad \pi_n(S^3) \simeq \pi_n(S^2) \text{ for } n \geq 3.$$